OPTIMAL ONE-DIMENSIONAL NON-LINEAR RESOURCE ALLOCATION*

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ABSTRACT

One-dimensional nonlinear resource allocation problems have been solved by the iterative use of a recurrence relation of the optimal state and decision for the system. Specifically two problems, the allocation of production effort to different facilities and a system reliability subject to a single nonlinear constraint, have considered. The solution of the first problem is obtained analytically; however, the solution of the second one is obtained numerically. The second problem illustrates a way of solving numerically a two-points boundary value problem which frequently occurs when the present method is used.

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1. INTRODUCTION

The distribution of effort problem is one in which a limited resource must be allocated among various activities. Such a problem arises in many situations, and the degree of difficulty of solution depends upon the form of the functions involved. Several people have solved the problem for specific types of functions. Koapmans (5) studied various increasing return functions and developed analytic solutions. The work was extended by Miehle (6) and numerical methods were developed for solutions of the additive return function. Charnes and Cooper (1) approximated the return function with piecewise linear continuous functions so that linear programming could be used. A general algorithm was developed by Karush (4) where the form of the return function was not restricted. However, he used a piecewise continuous linear approximation. Finally Shapiro and Wagner (7) solved the problem where the return function and the allocation function were both nonlinear and either convex or concave functions.

The present paper presents a fairly general approach to the distribution of effort problem with no restrictions on the return and allocation functions. Two problems, the allocation of production effort to different facilities and a system reliability subject to a nonlinear constraint, are studied.

2. RECURRENCE EQUATION FOR ONE-DIMENSIONAL RESOURCE ALLOCATION SYSTEMS

Suppose that a resource is to be allocated to N different activities. Also suppose that the objective is to maximize the total return from all the activities. If each activity is considered as a stage the resource allocation problem can be formulated as a multistage decision process as shown in Fig. 1. Let the allocation of a quantity R of the resource to the nth activity (stage) be θ^n and the return resulting from this allocation be $G^n(x_1^{n-1};\theta^n)$, where the state variable x_1^n is defined as

$$x_1^n = T^n(x_1^{n-1};\theta^n), \qquad n = 1, 2, \ldots, N, \tag{1}^*$$

$$x_1^0 = R, \tag{1a}$$

$$x_1^N = 0. \tag{1b}$$

In general, the objective function to be maximized is the sum of the return function over all stages of the system such as $\sum_{n=1}^{N} G^n(x_1^{n-1};\theta^n)$. If a new state variable is defined to satisfy

$$x_2^n = x_2^{n-1} + G^n(x_1^{n-1};\theta^n), \qquad n = 1, 2, \ldots, N, \tag{2}$$

$$x_2^0 = 0, \tag{2a}$$

it can be shown that

$$\sum_{n=1}^{N} G^n(x_1^{n-1};\theta^n) = x_2^N . \tag{3}$$

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*The superscript n indicates the stage number. The exponents are written with parenthesis or brackets as $(x^n)^2$ or $[G^n(x^{n-1};\theta^n)]^2$.

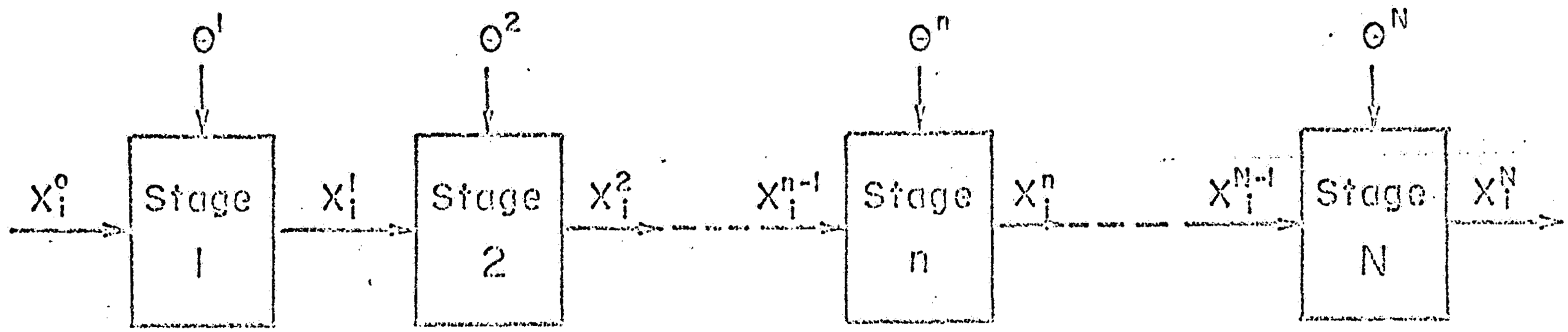

Fig. 1. Multistage decision process.

Thus the optimization problem of a one-dimensional resource allocation system can be formulated as a problem in which a set of θ^n, $n = 1, 2, \ldots, N$, is to be chosen to maximize x_2^N where

$$S = x_2^N \tag{4}$$

for a system described by equations (1) and (2).

It can be shown by a variety of means that the necessary (but not sufficient) condition for unconstrained local optimality can have the following recurrence representation (2,3).

$$\frac{\dfrac{\partial G^n(x_1^{n-1};\theta^n)}{\partial \theta^n}}{\dfrac{\partial T^n(x_1^{n-1};\theta^n)}{\partial \theta^n}} = \frac{\dfrac{\partial G^{n+1}(x_1^n;\theta^{n+1})}{\partial \theta^{n+1}}}{\dfrac{\partial T^{n+1}(x_1^n;\theta^{n+1})}{\partial \theta^{n+1}}} \frac{\partial T^{n+1}(x_1^n;\theta^{n+1})}{\partial x_1^n} - \frac{\partial G^{n+1}(x_1^n;\theta^{n+1})}{\partial x_1^n} , \tag{5}$$

$$n = 1, 2, \ldots, N-1 .$$

With x_1^N given, θ^N can be computed from equation (1) by assigning a value of x_1^{N-1}. The corresponding value for x_1^0 is obtained by iterative utilization of equations (1) and (5). The result is directly compared with the given x_1^0. The procedure is repeated until the computed value of x_1^0 is equal to the given value of x_1^0. It is worth noting that for each assigned value of x_1^{N-1}, the corresponding value of θ^n, $n = 1, 2, \ldots, N$, calculated are the optimal control actions corresponding to the initial condition x_1^0 computed in each run of trail calculations.

3. ALLOCATION OF PRODUCTION EFFORT TO DIFFERENT FACILITIES

A company must operate efficiently to succeed, and consequently, the available resources must be allocated so that its measure of efficiency is optimized. The resources and measure of success are usually the facilities and cost respectively. For a company with N production facilities which produce a single product, let n be the index for the appropriate facility and y^n its volume of output. The cost of producing y^n at each facility is

$$TC^n(y^n) = a^n + b^n y^n + c^n (y^n)^2 \tag{6}$$

where a^n, b^n, and c^n are positive constants.

The firm must produce exactly s units per time period and desires to split this production load between the N facilities so as to minimize the total product cost.

SOLUTION. Let each production facility represent a stage, and let

$\theta^n = y^n$ = volume of output at the n-th stage (n-th production facility),

x_1^n = volume of product which remains to be produced in (N-n) remaining stages,

x_2^n = total cost of production up to and including the n-th stage (production facility) where cost for the n-th stage is

$$TC^n(\theta^n) = a^n + b^n\theta^n + c^n(\theta^n)^2 . \tag{7}$$

Then the process may be described by the following two performance equations.

$$x_1^n = x_1^{n-1} - \theta^n; \qquad x_1^0 = s, \qquad x_1^N = 0, \tag{8}$$

$$x_2^n = x_2^{n-1} + [a^n + b^n (\theta^n) + c^n (\theta^n)^2], \quad x_2^0 = 0, \tag{9}$$

$n = 1, 2, \ldots, N.$

Comparing equations (8) and (9) with the performance equations, equations (1) and (2), we find

$$T^n(x_1^{n-1};\theta^n) = x_1^{n-1} - \theta^n, \tag{10}$$

$$G^n(x_1^{n-1};\theta^n) = a^n + b^n(\theta^n) + c^n(\theta^n)^2 . \tag{11}$$

Taking partial derivatives of equations (10) and (11) with respect to x_1^{n-1} and θ^n respectively, we obtain

$$\frac{\partial T^n(x_1^{n-1};\theta^n)}{\partial x_1^{n-1}} = 1, \tag{12}$$

$$\frac{\partial T^n(x_1^{n-1};\theta^n)}{\partial \theta^n} = -1, \tag{13}$$

$$\frac{\partial G^n(x_1^{n-1};\theta^n)}{\partial x_1^{n-1}} = 0, \tag{14}$$

and

$$\frac{\partial G^n(x_1^{n-1};\theta^n)}{\partial \theta^n} = b^n + 2\, c^n(\theta^n). \tag{15}$$

Substituting equations (12) through (15) into the recurrence equation,

equation (5), gives

$$\theta^n = \frac{b^{n+1} - b^n}{2\ c^n} + \frac{c^{n+1}}{c^n}\,\theta^{n+1}, \qquad n = 1, 2, \ldots, N-1. \tag{16}$$

Iterative use of the recurrence relations, equation (16) and equation (8), until the boundary conditions of equation (8) are satisfied gives the solution to this problem. However, a solution without iterative procedure can be obtained for this specific problem as below.

Let K be an assumed value of x_1^{N-1}. For n = N, equation (8)

$$x_1^{N-1} = \theta^N = K, \qquad (x_1^N = 0, \text{ given}). \tag{17}$$

From equations (16) and (17), we have

$$\theta^{N-1} = \frac{b^N - b^{N-1}}{2\ c^{N-1}} + \frac{c^N}{c^{N-1}}\,K, \quad (\theta^N = K). \tag{18}$$

Similarly

$$\theta^{N-2} = \frac{b^{N-1} - b^{N-2}}{2\ c^{N-2}} + \frac{c^{N-1}}{c^{N-2}}\,\theta^{N-1}\ . \tag{19}$$

Substituting equation (18) into equation (19) yields

$$\theta^{N-2} = \frac{b^N - b^{N-2}}{2\ c^{N-2}} + \frac{c^N}{c^{N-2}}\,K. \tag{20}$$

Continuing in this manner, we obtain the following expression for the decision at the n-th stage in terms of decision at the N-th stage.

$$\theta^n = \frac{b^N - b^n}{2\ c^n} + \frac{c^N}{c^n}\,K, \qquad (\text{where } \theta^N = K) \tag{21}$$

and for n = 1

$$\theta^1 = \frac{b^N - b^1}{2\ c^1} + \frac{c^N}{c^1}\ K\ . \tag{22}$$

Once the decision variables for all stages have been obtained, we calculate x_1^0 iteratively as follows. From equations (8) and (17), we obtain

$$\begin{aligned} x_1^{N-2} &= x_1^{N-1} + \theta^{N-1} \\ &= K + \frac{b^N - b^{N-1}}{2\ c^{N-1}} + \frac{c^N}{c^{N-1}}\ K. \end{aligned} \tag{23}$$

Combining equations (8) , (20), and (23) yields

$$\begin{aligned} x_1^{N-3} &= x_1^{N-2} + \theta^{N-2} \\ &= [K + \frac{c^N}{c^{N-1}} K + \frac{c^N}{c^{N-2}} K] + [\frac{b^N - b^{N-1}}{2\ c^{N-1}} + \frac{b^N - b^{N-2}}{2\ c^{N-2}}]\ . \end{aligned} \tag{24}$$

In general, we have

$$x_1^n = K\ c^N \left[\sum_{i=n+1}^{N} \frac{1}{c^i}\right] + \sum_{i=n+1}^{N-1} \left(\frac{b^N - b^i}{2\ c^i}\right)\ , \tag{25}$$

and in particular for $n = 0$, we obtain

$$x_1^0 = K\ c^N \left[\sum_{n=1}^{N} \frac{1}{c^n}\right] + \sum_{n=1}^{N-1} \left(\frac{b^N - b^n}{2\ c^n}\right)\ . \tag{26}$$

As $x_1^0 = s$, we can write

$$K\ c^N \left[\sum_{n=1}^{N} \frac{1}{c^n}\right] + \sum_{n=1}^{N-1} \left(\frac{b^N - b^n}{2\ c^n}\right) = s$$

or

$$K = \theta^N = \left[\frac{s - \sum_{n=1}^{N-1} \left(\frac{b^N - b^n}{2\,c^n}\right)}{\sum_{n=1}^{N} \frac{1}{c^n}} \right] \frac{1}{c^N} \quad . \tag{27}$$

Equations (16) and (27) give a complete sequence of decision variable θ^n; $n = 1, 2, \ldots, N$.

Let

$$\frac{s - \sum_{n=1}^{N-1} \frac{b^N - b^n}{2\,c^n}}{\sum_{n=1}^{N} \frac{1}{c^n}} = A \; . \tag{28}$$

Equation (27) then becomes

$$\theta^N = \frac{A}{c^N} \; . \tag{29}$$

Substituting equation (29) into equation (18) yields

$$\theta^{N-1} = \frac{b^N - b^{N-1}}{2\,c^{N-1}} + \frac{A}{c^{N-1}} \quad . \tag{30}$$

Substituting equation (29) into equation (20) gives

$$\theta^{N-2} = \frac{b^N - b^{N-2}}{2\,c^{N-2}} + \frac{A}{c^{N-2}} \quad .$$

By induction, it may be concluded that

$$\theta^n = \frac{b^N - b^n}{2\,c^n} + \frac{A}{c^n} \;, \qquad n = 1, 2, \ldots, N, \tag{31}$$

where A is given by equation (28).

It is worth mentioning that many problems of resource allocation belong to this class where the return function (cost or profit) is quadralic form as given by equation (11). It is obvious that the return function will be concave or convex if c^n is positive or negative, and accordingly the stationary point of the function gives a minimum or a maximum. Therefore, the recurrence equation given by equation (5) can be used if and only if the objective function is minimized (or maximized) when c^n of the return function given by equation (11) is positive (or negative). In other words, equation (5) gives the necessary condition of the optimality. This means that a policy or decision determined by use of equation (5) is not necessarily an optimal policy. Equation (5) only provides "a candidate or candidates" for the optimal policy. In general, the second order variation of the objective function around the candidate policy must be examined in order to determine if it is indeed the optimal policy. It is very difficult, if not impossible, to do so for any sort of a complex discrete system and we often have to resort to simulation or numerical search around the candidate policy.

4. SYSTEM RELIABILITY SUBJECT TO A SINGLE NONLINEAR CONSTRAINT

Consider a system with N-stages in series, each stage having some redundant units connected in parallel as shown in Fig. 2. The total number of units at each stage is subject to a separable non-linear constraint. The problem is to maximize the system reliability under this constraint.

Let us consider the constraint of the type [8],

$$g(\theta^n) = \sum_{n=1}^{N} \rho^n(\theta^n)^2 \leq P,$$

where ρ^n is constant for the nth stage and θ^n is the number of elements at the nth stage, therefore, number of redundancies employed at the nth stage is $(\theta^n - 1)$. An interpretation of this constraint is as follows

$$\rho^n(\theta^n)^2 = (w^n\theta^n)(c^n\theta^n),$$

where

w^n = weight per element at the nth stage and, therefore, $(w^n\theta^n)$ will be the total weight of the nth stage,

c^n = cost per element at the nth stage and, therefore, $(c^n\theta^n)$ will be the cost of the nth stage.

This leads to

$$\rho^n = c^n w^n \quad .$$

We define

x_1^n = amount of constraint left over after allocating first n stages,

x_2^n = logarithm of the reliability upto and including first n stages = $\ell n \prod_{i=1}^{n} (1 - (1-R^i)^{M^i})$,

where $(1-R^i)^{M^i}$ is the unreliability of the M^i components in parallel and each components has the reliability of R^i at the ith stage.

This N-stage system can now be presented by the following performance equations.

$$x_1^n = x_1^{n-1} - \rho^n(\theta^n)^2, \qquad n = 1, 2, \ldots, N, \tag{32}$$

$$x_1^0 = P, \tag{32a}$$

$$x_1^N \geq 0, \tag{32b}$$

$$x_2^n = x_2^{n-1} + \ell n \left(1 - (1 - R^n)^{\theta^n}\right), \; n = 1, 2, \ldots, N, \tag{33}$$

$$x_2^0 = 0. \tag{33a}$$

The objective function to be maximized is given by

$$S = \sum_{n=1}^{N} \ln \left(1 - (1 - R^n)^{\theta^n}\right) = x_2^N . \tag{34}$$

This corresponds to the logarithm of the maximum reliability.

Comparing equations (32) and (33) with equations (1) and (2) respectively, we have

$$T^n(x_1^{n-1},\theta^n) = x_1^{n-1} - \rho^n(\theta^n)^2 , \tag{35}$$

$$G^n(x_1^{n-1};\theta^n) = \ln \left(1 - (1 - R^n)^{\theta^n}\right) . \tag{36}$$

Taking partial derivatives of equations (35) and (36) with respect to x_1^{n-1} and θ^n respectively, we obtain

$$\frac{\partial T^n(x_1^{n-1};\theta^n)}{\partial x_1^{n-1}} = 1, \tag{37a}$$

$$\frac{\partial T^n(x_1^{n-1};\theta^n)}{\partial \theta^n} = -2\rho^n\theta^n, \tag{37b}$$

$$\frac{\partial G^n(x_1^{n-1};\theta^n)}{\partial x_1^{n-1}} = 0, \tag{37c}$$

$$\frac{\partial G^n(x_1^{n-1};\theta^n)}{\partial \theta^n} = \frac{-(1-R^n)^{\theta^n} \ln(1-R^n)}{1 - (1-R^n)^{\theta^n}} . \tag{37d}$$

Substituting equations (37a) through (37d) into the recurrence equation, equation (5), gives

$$\frac{\dfrac{(U^n)^{\theta^n}\,\ln(U^n)}{\left(1-(U^n)^{\theta^n}\right)}}{-2\rho^n\theta^n} = \frac{\dfrac{(U^{n+1})^{\theta^{n+1}}\,\ln(U^{n+1})}{\left(1-(U^{n+1})^{\theta^{n+1}}\right)}}{-2\rho^{n+1}\theta^{n+1}} .$$

where $U^n = 1 - R^n$. Rearranging the terms, this equation yields

$$\theta^{n+1} = \left(A^n + \theta^{n+1}\right)\,(U^{n+1})^{\theta^{n+1}} , \tag{38}$$

where

$$A^n = \left[\frac{\rho^n}{\rho^{n+1}}\right]\left[\frac{\ln U^{n+1}}{\ln U^n}\right]\left[\frac{1-(U^n)^{\theta^n}}{(U^n)^{\theta^n}}\right]\theta^n .$$

Note that all terms of A^n are known if θ^n is known. Therefore equation (38) can be solved by Newton's method. Also note that, while θ^n, $n = 1, 2, \ldots, N$ are in reality positive integers it is assumed that θ^n are continuous variables in obtaining the above recurrence relation. The above equation may be written as

$$f(\theta^{n+1}) = \theta^{n+1} - (A^n + \theta^{n+1})(U^{n+1})^{\theta^{n+1}} = 0 . \tag{39}$$

As U^{n+1} is less than one, it can be seen that $f(\theta^{n+1})$ will be a monotonically increasing function of θ^{n+1}. At $\theta^{n+1} = 0$ we find that $f(\theta^{n+1})$ is negative. As we increase the value of θ^{n+1}, the value of this function increases. Therefore, starting from a negative value, $f(\theta^{n+1})$ will pass through zero and then become positive. The stepwise procedure is given below.

COMPUTATIONAL PROCEDURE

Step 1. Assume $\theta^1 = 1$, and $n = 1$.

Step 2. Assume $(\theta^{n+1})_1 = 1$, where $(\theta^{n+1})_1$ is the first trial solution of equation (39).

Step 3. Compute A^n.

Step 4. Compute

$$f\{(\theta^{n+1})_1 = (\theta^{n+1})_1 - \left(A^n + (\theta^{n+1})_1\right) (U^{n+1})^{(\theta^{n+1})_1} ,$$

and

$$f'\{(\theta^{n+1})_1\} = 1 - \left((U^{n+1})^{(\theta^{n+1})_1} + \{(\theta^{n+1})_1 + A^n\}(U^{n+1})^{(\theta^{n+1})_1} \ln U^{n+1}\right). \tag{40}$$

Step 5. Compute a new trial value for θ^{n+1} from following equation

$$(\theta^{n+1})_2 = (\theta^{n+1})_1 - \frac{f\left((\theta^{n+1})_1\right)}{f'\left((\theta^{n+1})_1\right)} .$$

Step 6. Check if

$$|(\theta^{n+1})_2 - (\theta^{n+1})_1| \leq (E_R)_{max} .$$

$(E_R)_{max}$ will depend on the accuracy of the result desired. If this is satisfied consider that $(\theta^{n+1})_2$ is an optimum solution corresponding to the assumed value of θ^1, go to step 7, if this is not satisfied go to step 4 with the new value of θ^{n+1}, that is, replacing $(\theta^{n+1})_1$ by $(\theta^{n+1})_2$.

Step 7. Increase n by one and go to step 2 until n becomes greater than N; when n becomes greater than N go to step 8.

Step 8. Compute

$$x_1^N = P - \sum_{r=1}^{N} \rho^n(\theta^n)^2 .$$

This give rise to one of the following conditions. (a) x_1^N is greater than zero, (b) equal to zero and (c) less than zero.

If it is (a), go to step 9, if it is (b), we have an optimal solution, round off the solution to integers, if it is (c), go to step 10.

Step 9. Increment θ^1 by one, make $n = 1$ again and go back to step 3.

Step 10. Reduce θ^1 by 0.1, make $n = 1$ again and go to step 3 until x_1^N again becomes greater than zero. When x_1^N is greater than zero the solution is optimal one. Round off the solution to an integer.

The procedures are illustrated in the flow chart (Fig. 3).

NUMERICAL EXAMPLE

An eight stage problem is solved for illustration. However, the numerical method developed are for any system with an arbitrary number of stages.

The constants employed for this illustrative problem are given in Table 1. The optimum redundancy obtained is as follows.

$$\theta^1 = 2.30,$$
$$\theta^2 = 1.57,$$
$$\theta^3 = 3.31,$$
$$\theta^4 = 5.33,$$
$$\theta^5 = 3.25,$$
$$\theta^6 = 3.63,$$
$$\theta^7 = 3.15,$$
$$\theta^8 = 2.53.$$

Since θ^n, $n = 1, 2, \ldots, 8$, in reality, should be positive integers, we have

Fig. 3. Computer flow chart.

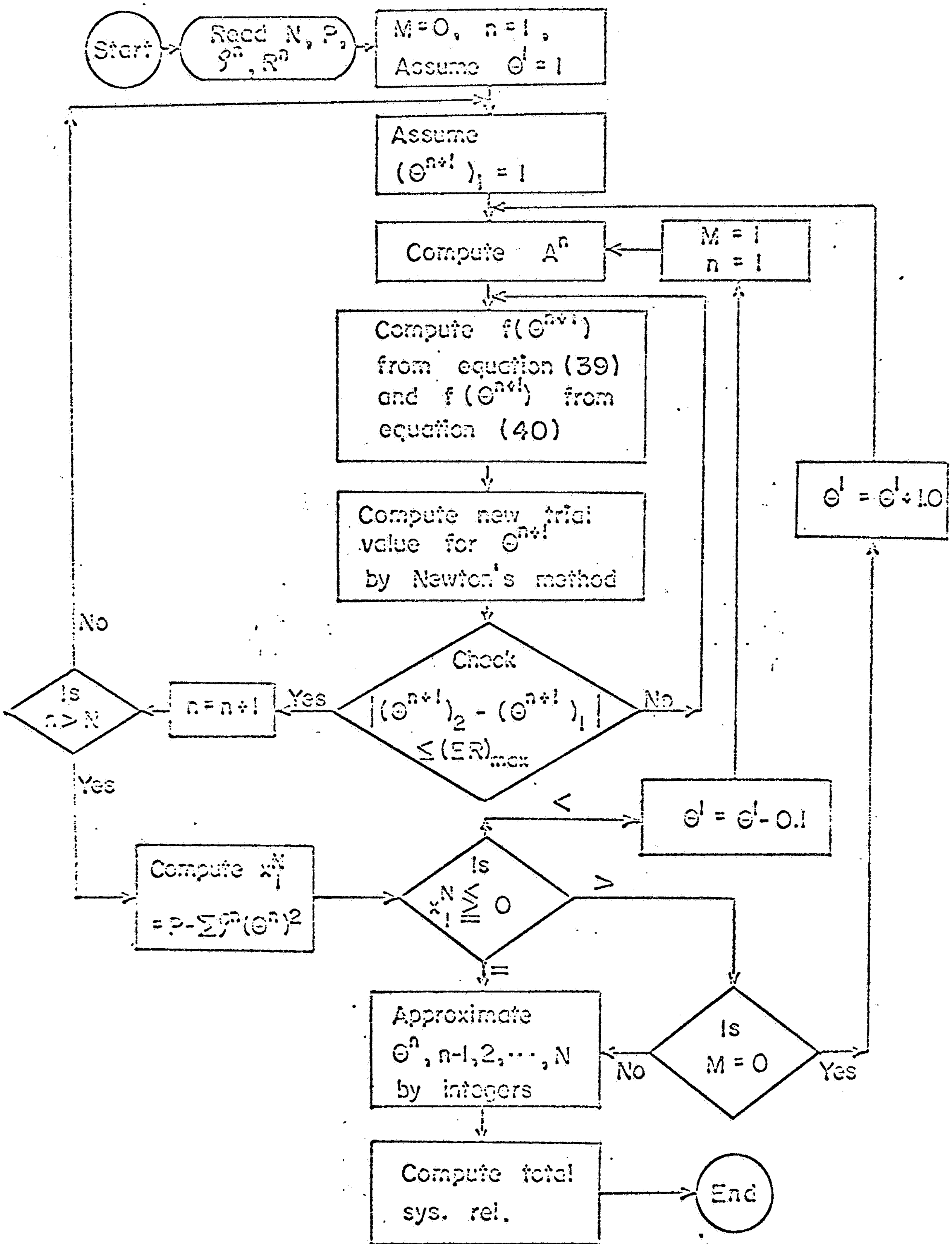

Table 1. Constants assigned for 8 stage problem.

n	R^n	ρ^n	P
1	.85	5.00	
2	.95	8.00	
3	.75	2.00	
4	.55	1.00	
5	.60	7.00	
6	.65	3.00	
7	.70	4.00	
8	.80	5.00	
			300

$$\theta^1 = 2,$$

$$\theta^2 = 2,$$

$$\theta^3 = 3,$$

$$\theta^4 = 5,$$

$$\theta^5 = 3,$$

$$\theta^6 = 4,$$

$$\theta^7 = 3,$$

$$\theta^8 = 3.$$

Since θ^n represents the number of elements at each stage, the number of redundant units at each stage is obtained by subtracting one from each of them.

Here we note that 287 units out of 300 units of resource is used which gives the optimum reliability of 0.8384. Results of numerical simulation indicates the above result is not significantly different from the truly optimal one.

REFERENCES

1. Charnes, A. and W. W. Cooper, "The Theory of Search: Optimum Distribution of Search Effort", Management Science, Vol. 5, pp. 44-50, 1958.

2. Denn, M. M. and R. Aris, "Generalized Euler Equations", J. of Applied Math. and Physics (ZAMP), Vol. 16, pp. 290-295 (1965).

3. Fan, L. T. and C. S. Wang, The Discrete Maximum Principle--A Study of Multistage Systems Optimization, Wiley, New York, 1964.

4. Karush, W., "A General Algorithm for the Optimal Distribution of Effort", Management Science, Vol. 9, pp. 50-72, 1962.

5. Koopman, B. O., "The Optimum Distribution of Effort", Operations Research, Vol. 1, pp. 52-63, 1952.

6. Miehle, W., "Numerical Solution of the Problem of Optimum Distribution of Effort", Operations Research, Vol. 2, pp. 433-440, 1954.

7. Shapiro, J. F. and H. M. Wagner, "A Finite Revewal Algorithm for the Knapsack and Turnpike Models", Operations Research, Vol. 15, pp. 319-341, 1967.

8. Tillman, F. A. and J. M. Liittschwager, "Integer Programming Formulation of Constrained Reliability Problems", Management Science, Vol. 13, pp. 887-889, 1967.